An Approximation of the Outage Probability for Multi-hop AF Fixed Gain Relay

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***Abstract*— In this letter, we present a closed-form approximation of the outage probability for the multi-hop amplify-and-forward (AF) relaying systems with fixed gain in Rayleigh fading channel. The approximation is derived from the outage event for each hop. The simulation results show the tightness of the proposed approximation in low and high signal-to-noise ratio (SNR) region.**

***Index Terms*— Wireless relay channel, multi-hop relay, outage probability**

I. Introduction

OUTAGE probability is an important measure of system performances in fading channel for reliability of link quality. Although multihop AF fixed gain relays can be more preferable for simple implementation, due to complication in analysis associated with noise accumulation factor with multi-hop relay, there are not many existing theoretical researches on them. In [1] and [2], the outage probabilities of multihop decode-and-forward (DF) relay systems were calculated and the optimum power allocation schemes were proposed. In [1], it is argued that the outage probability of multihop DF relay systems can be the lower bound of multihop AF relay systems. In [3], Hasna et. al. found the outage probability for multihop AF variable gain relay systems. In [4], Karagiannidis provided the bounds of the outage probability for multihop AF fixed gain relay systems by using harmonic and geometric mean, which are not tight in high SNR region.

In this letter, we derive a closed-form approximation of the outage probability for multihop AF fixed gain relay systems by a novel approach considering the event space of the outage, which is tight in all SNR regions. The remainder of this letter is organized as follows. The system and channel model for the multihop AF fixed gain relay system is presented, and the received SNR at the destination is derived in Section II. In Section III, outage probability of relaying system is derived for arbitrary number of hops. In Section IV, we discuss the theoretical results and the simulation results for the system. Finally, we conclude this letter in Section V.

II. System model

Considering the general N-hop relay network, there are $(N-1)$ relays between the source and the destination. It is assumed that the relaying network is operated on time division multiplexing (TDM) so that the transmission at any node occurs in different time slots on the same carrier frequency.

Assuming that a signal at the source is transmitted with an average power E_1 and the fixed gain relays are serially placed from the source to the destination, the instantaneous end-to-end SNR at the destination can be written as

$$\gamma_N = \frac{A_{N-1}^2 A_{N-2}^2 \cdots A_1^2 |h_N|^2 |h_{N-1}|^2 \cdots |h_1|^2 E_1}{\sum_{j=1}^{N-1} \left(\prod_{i=j}^{N-1} A_i^2 |h_{i+1}|^2 \right) \sigma_j^2 + \sigma_N^2} \quad (1)$$

where h_k is the fading amplitude of the channel at the k-th hop with unit variance, i.e., $E\{|h_k|^2\} = 1$ where $k = 1, \cdots, N$, in which $E\{\cdot\}$ is the expectation operator, and σ_k^2 is the variance of the additive white Gaussian noise (AWGN) with mean zero at the k-th hop.

In (1), the amplification factor of the l-th relay with fixed gain is defined [5] as

$$A_l = \sqrt{\frac{E_{l+1}}{E_l + \sigma_l^2}} \;,\; l = 1, \cdots, N-1 \quad (2)$$

III. Outage probability for multi-hop relaying systems with fixed gain

The outage probability is defined as the probability that the end-to-end SNR, γ_N , falls below a threshold level of SNR. For N-hop AF fixed gain relay system, the outage probability can be written as

$$P_{out}(\gamma_N < \gamma_{Th}) \quad (3)$$

where γ_{Th} is the threshold level of SNR.

Unfortunately, it is very difficult to obtain the closed-form expression for the outage probability of multi-hop AF fixed gain relay system due to the noise accumulation, as shown in the denominator of (1). Alternatively, the outage probability of DF relay in [1] or of AF variable gain relay in [3] were proposed to be lower bounds for AF fixed gain relay. However, those can be definitely lower bounds but are not tight. [4] found a lower bound for AF fixed gain relay by using the well-known inequality between harmonic and geometric mean. But it loses tightness in high SNR region. In this letter, rather than finding a bound, we focus on finding out a closed-form accurate approximation of the outage probability for multihop AF fixed gain relay. To this end, the following theorem is introduced first.

Theorem 1: The outage probability for the N-hop AF relaying system with fixed gain is lower bounded by $G_{1,N+1}^{N,1}\left[\bar{\gamma}_{Th} \,\middle|\, \begin{matrix} 1 \\ 1,\ 1,\ \cdots,\ 1, 0 \end{matrix}\right]$, where $\bar{\gamma}_{Th} = \frac{\sigma_N^2}{A_{N-1}^2 A_{N-2}^2 \cdots A_1^2 E_1} \gamma_{Th}$.

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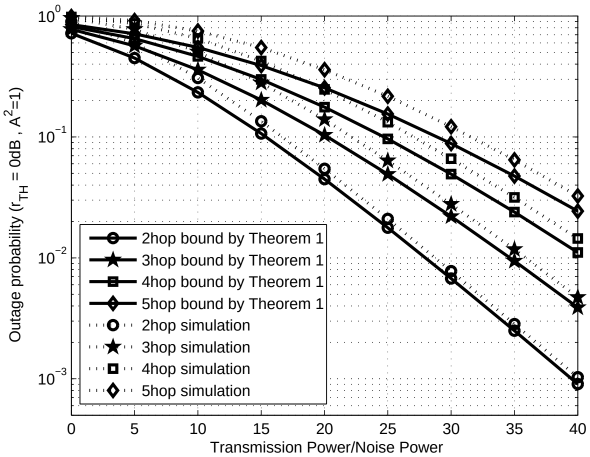

Fig. 1. The bound in Theorem 1 of outage probability for multihop AF relay system with fixed gain ($A_1^2 = \cdots = A_{N-1}^2 = 1$).

Proof: The end-to-end SNR of N-hop AF relay is upper bounded to $\tilde{\gamma}_N$ with the assumption that relays operate at asymptotically high SNR, i.e., $\sigma_1^2 = \ \cdots \ = \sigma_{N-1}^2 = 0$.

$$\gamma_N \leq \frac{A_{N-1}^2 A_{N-2}^2 \cdots A_1^2 \left|h_N\right|^2 \left|h_{N-1}\right|^2 \ \cdots \ \left|h_1\right|^2 E_1}{\sigma_N^2} \triangleq \tilde{\gamma}_N \tag{4}$$

From (4), the outage probability is obviously lowered bounded as

$$P_{out}(\tilde{\gamma}_N < \gamma_{Th}) \leq P_{out}(\gamma_N < \gamma_{Th}) \tag{5}$$

The left term of (5) can be equivalently expressed as

$$P_{out}(\gamma'_N < \bar{\gamma}_{Th,N}) \leq P_{out}(\gamma_N < \gamma_{Th}) \tag{6}$$

where $\gamma'_N = \left|h_1\right|^2 \left|h_2\right|^2 \cdots \left|h_N\right|^2$ and $\bar{\gamma}_{Th,N} = \frac{\sigma_N^2}{A_{N-1}^2 A_{N-2}^2 \cdots A_1^2 E_1}\gamma_{Th}$.

By using Weibull distribution which is the general form of the exponential distribution family, the PDF of the cascaded exponential random variables, $\gamma = \left|h_1\right|^2 \left|h_2\right|^2 \ \cdots \ \left|h_N\right|^2$, can be expressed by using [6, eq. (3)] as

$$f_\gamma(\gamma) = \frac{1}{\gamma} G_{N,0}^{0,N}\left[\frac{1}{\gamma}\middle| \begin{array}{c} 0,\ 0,\ \cdots\ ,\ 0 \\ - \end{array}\right] \tag{7}$$

where $G\left[\cdot\right]$ is the Meijer G function in [7, eq. (9.301)] which is a built-in function in the well-known softwares such as MAPLE and MATHEMATICA.

The lower bound of the outage probability for the N-hop AF fixed gain relay can be calculated with the help of [8, eq. (07.34.16.0002.01)] and [8, eq. (07.34.21.0001.01)] as

$$\begin{aligned} & P_{out}(\tilde{\gamma}_N < \gamma_{Th}) = P_{out}(\gamma'_N < \bar{\gamma}_{Th,N}) \\ & = \int_0^{\bar{\gamma}_{Th,N}} \tfrac{1}{\gamma} G_{N,0}^{0,N}\left[\tfrac{1}{\gamma}\middle| \begin{array}{c} 0,\ 0,\ \cdots\ ,\ 0 \\ - \end{array}\right] d\gamma \\ & = G_{1,N+1}^{N,1}\left[\bar{\gamma}_{Th,N}\middle| \begin{array}{c} 1 \\ 1,\ 1,\ \cdots\ ,\ 1, 0 \end{array}\right] \end{aligned} \tag{8}$$

■

As shown in Fig. 1, Theorem 1 is confirmed as a lower bound of outage probability for multihop AF fixed gain relay by the simulation, assuming that all amplification factors of the relays are 1. However, (8) loses tightness in low SNR region, since the closed-form lower bound is derived under the high SNR assumption. Therefore, the different method should be considered to increase accuracy in low SNR region.

Fig. 2. The outage event space for multihop AF relaying system with fixed gain.

Lemma 1: In N-hop AF fixed gain relay network, as the number of hops is increased, the end-to-end SNR is decreased, $\gamma_N \leq \gamma_{N-1} \leq \ \cdots \ \leq \gamma_2 \leq \gamma_1$, while outage probability is increased, $P_{out}(\gamma_1 < \gamma_{Th}) \leq P_{out}(\gamma_2 < \gamma_{Th}) \leq \cdots \leq P_{out}(\gamma_N < \gamma_{Th})$.

Proof: It can be easily proved. ■

For notational simplicity, let $P_{out,n}$ and $P_{out,n}^*$ for $n = 1, 2, \cdots N$ be denoted by $P_{out}(\gamma_n < \gamma_{Th})$ and $P_{out}(\tilde{\gamma}_n < \gamma_{Th})$, respectively. From Theorem 1 and Lemma 1, the outage event space for multihop AF relaying system with fixed gain can be drawn as Fig. 2.

As shown in Fig. 2, the outage probability can be expressed as the sum of the probabilities for each hop. The outage probability can be evaluated for each number of hops in the following way.

(1) 1-hop case:

$$P_{out,1} = P(\gamma_1 < \gamma_{Th}) = P_{out}(\gamma'_1 < \bar{\gamma}_{Th,1}) = 1 - e^{-\bar{\gamma}_{Th,1}} \tag{9}$$

(2) 2-hop case:

$$\begin{aligned} & P_{out,2} = P(\gamma_2 < \gamma_{Th}) \\ & = P(\gamma_1 < \gamma_{Th},\ \gamma_2 < \gamma_{Th}) + P(\gamma_1 > \gamma_{Th},\ \gamma_2 < \gamma_{Th}) \end{aligned} \tag{10}$$

where the outage probability cannot be directly calculated since γ_1 and γ_2 are not independent. However, it can be approximated with the help of Fig. 2. The first term in (10) is obviously rewritten as $P(\gamma_1 < \gamma_{Th})$ and the second term is approximately calculated as $P(\gamma_1 > \gamma_{Th})P(\gamma_2 < \gamma_{Th})$ for simplicity, as if the two events are independent. Since it is hard to have a closed form expression for $P(\gamma_2 < \gamma_{Th})$, it can approximated by using the lower bound in Lemma 1.

Therefore, the approximation of the outage probability for the 2-hop case can be obtained as

$$
\begin{aligned}
P_{out,2} &\approx P(\gamma_1' < \bar{\gamma}_{Th,1}) + P(\gamma_1' > \bar{\gamma}_{Th,1})P(\gamma_2 < \gamma_{Th}) \\
&\geq P_{out,1} + P(\gamma_1' > \bar{\gamma}_{Th,1})P(\gamma_2' < \bar{\gamma}_{Th,2}) \triangleq \tilde{P}_{out,2}
\end{aligned} \tag{11}
$$

(3) 3-hop case:

$$
\begin{aligned}
&P_{out,3} = P(\gamma_3 < \gamma_{Th}) \\
&= P(\gamma_1 < \gamma_{Th},\ \gamma_2 < \gamma_{Th}, \gamma_3 < \gamma_{Th}) \\
&\quad + P(\gamma_1 > \gamma_{Th},\ \gamma_2 < \gamma_{Th}, \gamma_3 < \gamma_{Th}) \\
&\quad + P(\gamma_1 > \gamma_{Th},\ \gamma_2 > \gamma_{Th}, \gamma_3 < \gamma_{Th}) \\
&\approx \tilde{P}_{out,2} + P(\gamma_3 < \gamma_{Th})P(\gamma_1 > \gamma_{Th},\ \gamma_2 > \gamma_{Th}) \\
&\approx \tilde{P}_{out,2} + P(\gamma_3' < \bar{\gamma}_{Th,3})P(\gamma_3 > \gamma_{Th}) \\
&= \tilde{P}_{out,2} + P(\gamma_3' < \bar{\gamma}_{Th,3})(1 - P(\gamma_3 < \gamma_{Th})) \\
&\approx \tilde{P}_{out,2} + P(\gamma_3' < \bar{\gamma}_{Th,3})(1 - P(\gamma_3' < \bar{\gamma}_{Th,3})) \triangleq \tilde{P}_{out,3}
\end{aligned} \tag{12}
$$

Similarly, for general N-hop AF relaying system with fixed gain, its outage probability can be approximated as

(4) N-hop case:

$$
\begin{aligned}
&P_{out,N} = P(\gamma_N < \gamma_{Th}) \\
&\approx \tilde{P}_{out,N-1} + P(\gamma_N' < \bar{\gamma}_{Th,N})(1 - P(\gamma_N' < \bar{\gamma}_{Th,N})) \\
&\quad \triangleq \tilde{P}_{out,N}
\end{aligned} \tag{13}
$$

To increase the accuracy of the approximation, we perform averaging out the noise power in (1) and applying it to the right term in (13). If $\sigma_1^2 = \cdots = \sigma_N^2$, then the approximation of the outage probability can be written as

$$
\begin{aligned}
P_{out,N} &\approx \tilde{P}_{out,N-1} + P\left(\gamma_N' < \bar{\gamma}_{Th,N}\right) \\
&\cdot \left(1 - P\left(\gamma_N' < \left[\sum_{j=1}^{N-1}\left(\prod_{i=j}^{N-1} A_i^2\right) + 1\right]\bar{\gamma}_{Th,N}\right)\right)
\end{aligned} \tag{14}
$$

Without loss of generality, for $N \geq 3$, the outage probability in (14) can be rewritten as

$$
\begin{aligned}
&P_{out,N} \\
&\approx (1 - e^{-\bar{\gamma}_{Th,1}}) + e^{-\bar{\gamma}_{Th,1}} G_{1,3}^{2,1}\left[\bar{\gamma}_{Th,2}\,\middle|\,\begin{matrix} 1 \\ 1,\ 1,\ 0 \end{matrix}\right] \\
&+ \sum_{n=3}^{N} G_{1,n+1}^{n,1}\left[\bar{\gamma}_{Th,n}\,\middle|\,\begin{matrix} 1 \\ 1,\ 1,\ \cdots,\ 1, 0 \end{matrix}\right]\Bigg(1 - \\
&G_{1,n+1}^{n,1}\left[\left(\sum_{j=1}^{n-1}\left(\prod_{i=j}^{n-1} A_i^2\right) + 1\right)\bar{\gamma}_{Th,n}\,\middle|\,\begin{matrix} 1 \\ 1, 1, \cdots, 1, 0 \end{matrix}\right]\Bigg)
\end{aligned} \tag{15}
$$

IV. Simulation results

The proposed approximation and the simulation results of the outage probability for multihop AF relaying system with fixed gain are shown in Fig. 3. For the simulation, the threshold level of SNR is 0dB. From the results, it is noted that the proposed approximation of the outage probability is very close to the simulation result in low and high SNR region.

V. Conclusion

In this letter, an accurate approximation of the outage probability for the multihop AF relaying systems with fixed gain in Rayleigh fading channel is derived by using the outage event space. The numerical results show that it provides the very accurate approximation in all SNR region.

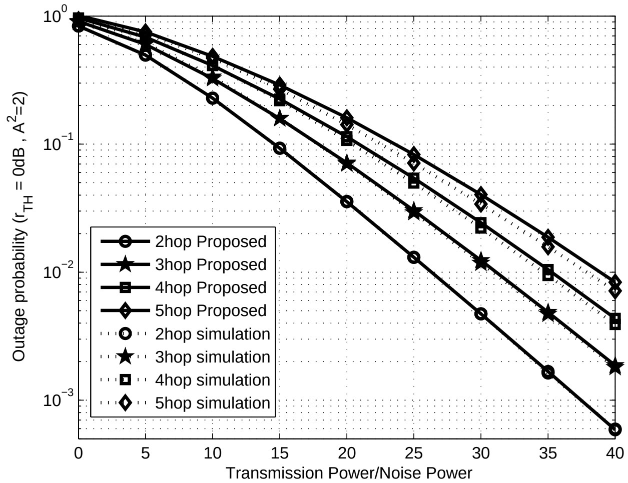

Fig. 3. The proposed approximation of outage probability for multihop AF relay system with fixed gain ($A_1^2 = \cdots = A_{N-1}^2 = 2$).

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